4.5 Substitution:

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then $\boxed{\int f(g(x))\, g'(x)\, dx = \int f(u)\, du}$

In case of definite integrals: $\boxed{\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du}$

① $\int x^4 (3+5x^5)^4 \, dx$ $\quad \boxed{u = 3+5x^5}$ because $\boxed{du = 25x^4\, dx}$

$\Rightarrow \boxed{x^4 dx = \frac{1}{25} du}$

$$\int x^4 \overbrace{(3+5x^5)^4}^{u^4} dx$$

$\frac{1}{25} du$

$$\int u^4 \frac{1}{25} du = \frac{1}{25} \frac{u^5}{5} + C = \frac{u^5}{125} + C$$

$$= \frac{(3+5x^5)^5}{125} + C$$

② $\int \frac{x^4}{\sqrt{6+10x^5}} dx$ $\quad \boxed{u = 6+10x^5} \Rightarrow \boxed{du = 50x^4 dx} \Rightarrow \boxed{x^4 dx = \frac{1}{50} du}$

$\frac{1}{50} du$

$$\int \frac{x^4}{\underbrace{\sqrt{6+10x^5}}_{\sqrt{u}}} dx = \int \frac{1}{50} \frac{1}{\sqrt{u}} du = \frac{1}{25}\sqrt{u} + C$$

$$= \frac{1}{25}\sqrt{6+10x^5} + C$$

③ $\int \cos^{12}(5t) \sin(5t) \, dt$ $\quad \boxed{u = \cos(5t)} \Rightarrow \boxed{du = -5\sin(5t)\, dt}$

$\Rightarrow \boxed{\sin(5t)\, dt = -\frac{1}{5} du}$

$$\int u^{12} \left(-\frac{1}{5}\right) du = -\frac{1}{5} \frac{u^{13}}{13} + C$$

$$= -\frac{1}{5} \frac{\cos^{13}(5t)}{13} + C$$

④ $\int \sqrt{10-2x}\,dx$ $\quad \boxed{u = 10-2x} \Rightarrow \boxed{du = -2dx} \Rightarrow \boxed{dx = -\frac{1}{2}du}$

$$= \int \sqrt{u}\left(-\frac{1}{2}\right)du = -\frac{1}{2}\,\frac{u^{3/2}}{3/2} = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(10-2x)^{3/2} + C$$

⑤ $\int \frac{6\sin(\sqrt{x})}{\sqrt{x}}\,dx$ $\quad \boxed{u=\sqrt{x}} \Rightarrow \boxed{du = \frac{1}{2\sqrt{x}}dx} \Rightarrow \boxed{\frac{1}{\sqrt{x}}dx = 2du}$

$$= \int 6\sin(u)\cdot 2du = 12\int \sin(u)\,du = -12\cos(u)+C$$

$$= -12\cos(\sqrt{x})+C$$

⑥ $\int \frac{1}{x^2}\sin\left(\frac{2}{x}\right)\cos\left(\frac{2}{x}\right)dx$

$\boxed{u = \sin\left(\frac{2}{x}\right)} \Rightarrow \boxed{du = \cos\left(\frac{2}{x}\right)\cdot\frac{-2}{x^2}\,dx}$

$\Rightarrow \boxed{\frac{1}{x^2}\cos\left(\frac{2}{x}\right)dx = -\frac{1}{2}du}$

$$\int u\cdot\frac{-1}{2}du = -\frac{1}{2}\int u\,du = -\frac{1}{2}\,\frac{u^2}{2}+C = -\frac{1}{4}u^2+C$$

$$= -\frac{1}{4}\sin^2\left(\frac{2}{x}\right)+C$$

Another way:

$\boxed{u = \cos\left(\frac{2}{x}\right)} \Rightarrow \boxed{du = -\sin\left(\frac{2}{x}\right)\cdot\frac{-2dx}{x^2} = \frac{2}{x^2}\sin\left(\frac{2}{x}\right)dx}$

$\Rightarrow \boxed{\frac{1}{x^2}\sin\left(\frac{2}{x}\right) = \frac{1}{2}du}$

$$\Rightarrow \int \underbrace{\frac{1}{x^2}\sin\left(\frac{2}{x}\right)dx}_{\frac{1}{2}du}\,\underbrace{\cos\left(\frac{2}{x}\right)}_{u} = \int u\,\frac{1}{2}du = \frac{1}{2}\int u\,du = \frac{1}{2}\,\frac{u^2}{2}+C$$

$$= \frac{1}{4}\cos^2\left(\frac{2}{x}\right)+C$$

These two answers are equivalent!

$$\frac{1}{4}\cos^2\left(\frac{2}{x}\right)+C = \frac{1}{4}\left(1-\sin^2\left(\frac{2}{x}\right)\right)+C = -\frac{1}{4}\sin^2\left(\frac{2}{x}\right)+\frac{1}{4}+C$$

$$\Rightarrow +C$$

7. $\int \sec^2(4x)\tan^2(4x)\,dx$

$$\boxed{u = \tan(4x)} \Rightarrow \boxed{du = 4\sec^2(4x)\,dx} \Rightarrow \boxed{\sec^2(4x)\,dx = \frac{1}{4}du}$$

$$= \int u^2 \frac{1}{4}du = \frac{1}{4}\frac{u^3}{3} + C = \frac{1}{12}\tan^3(4x) + C$$

8. $\int 8x\sqrt{x-4}\,dx$

$$\boxed{u = x-4} \Rightarrow \boxed{du = dx} \Rightarrow x = u+4$$

$$\int 8(u+4)\sqrt{u}\,du = 8\int\left(u^{3/2} + 4u^{1/2}\right)du = 8\left(\frac{u^{5/2}}{5/2} + 4\frac{u^{3/2}}{3/2}\right) + C$$

$$= \frac{16}{5}u^{5/2} + \frac{64}{3}u^{3/2} + C = \frac{16}{5}(x-4)^{5/2} + \frac{64}{3}(x-4)^{3/2} + C$$

9. $\int_{-7}^{2} \frac{1}{\sqrt{13-4x}}\,dx$

$u = 13-4x$ $\Rightarrow$ when $x=-7$, $u = 13+28 = 41$

$du = -4\,dx$ when $x = 2$, $u = 13-8 = 5$

$dx = -\frac{1}{4}du$

$$= \int_{41}^{5} \frac{1}{\sqrt{u}}\,\frac{-1}{4}\,du = \int_{5}^{41}\frac{1}{4\sqrt{u}}\,du = \frac{1}{2}\sqrt{u}\,\Big|_5^{41} = \frac{1}{2}\left(\sqrt{41}-\sqrt{5}\right)$$

10. $\int_0^{\pi/30} 15\tan(5x)\sec^2(5x)\,dx$ $= \int_1^{2/\sqrt{3}} 3u\,du = \frac{3u^2}{2}\Big|_1^{2/\sqrt{3}}$

$u = \sec(5x)$

$du = \sec(5x)\tan(5x)\cdot 5\,dx$

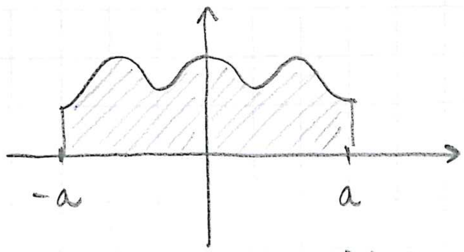
When $x=0$, $u = \sec(0) = 1$

When $x = \pi/30$, $u = \sec(\pi/6) = \frac{2}{\sqrt{3}}$

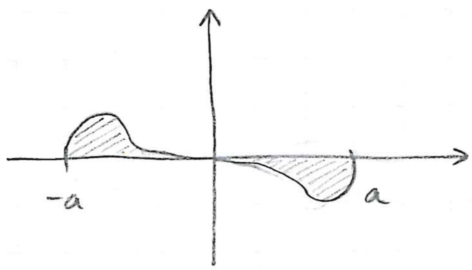
$$= \cancel{\frac{3}{2}}\cdot\frac{\overset{2}{\cancel{4}}}{\cancel{3}} - \frac{3}{2} = 2 - \frac{3}{2} = \frac{1}{2}$$

Symmetry:

* If f is <u>even</u> on $[-a,a]$, i.e. $f(-x)=f(x)$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
  ↳ symmetric about the y-axis



* If f is <u>odd</u> on $[-a,a]$, i.e. $f(-x)=-f(x)$, then $\int_{-a}^{a} f(x)\,dx = 0$
  ↳ symmetric about the origin



<u>Example:</u> $\int_{-6}^{6} \frac{t^3}{19+9t^2}\,dt = 0$ because $\frac{t^3}{19+9t^2}$ is an <u>odd</u> function.

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(11) $\int x^3\cos(x^4+2)\,dx$

$u = x^4+2$
$du = 4x^3\,dx$
$x^3\,dx = \frac{1}{4}\,du$

$\int \cos(u)\,\frac{1}{4}\,du = \frac{1}{4}\sin(u)+C$
$= \frac{1}{4}\sin(x^4+2)+C.$

(12) $\int_{1}^{2} \frac{dx}{(3-5x)^2}$

$u = 3-5x$
$du = -5\,dx$
$dx = -\frac{1}{5}\,du$

$x=1 \Rightarrow u = 3-5 = -2$
$x=2 \Rightarrow u = 3-10 = -7$

$\int_{-2}^{-7} \frac{1}{u^2}\,\frac{-1}{5}\,du = \left.\frac{1}{5u}\right|_{-2}^{-7} = \frac{1}{5}\left(\frac{-1}{7}+\frac{1}{2}\right) = \frac{1}{5}\cdot\frac{5}{14} = \frac{1}{14}.$